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A Characterization of Entire Functions $\sum_{k=0}^{\infty} a_k z^k$ with all $a_k \geqslant 0$

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THEOREM. *Let a function f with domain* $[0, \infty)$ *be positive and continuous there. A necessary and sufficient condition for the existence of a sequence* $(p_n(x))_{n=0}^{\infty}$ *of polynomials whose coefficients are* $\geqslant 0$, *all* $p_n(0) > 0$, *satisfying*

$$\sup_{0\leqslant x<\infty} \left| \frac{1}{f(x)} - \frac{1}{p_n(x)} \right| \to 0 \qquad as \quad n \to \infty$$

is that f be the restriction of an entire function $\sum_{k=0}^{\infty} a_k z^k$, *with all* $a_k \geqslant 0$.

Proof. Sufficiency. For $n = 0, 1, 2,\ldots$, set $p_n(z) \equiv \sum_{k=0}^{n} a_k z^k$, so that $p_n(0) = a_0 = f(0) > 0$. Let $\epsilon > 0$. We may assume some $a_k (k \geqslant 1)$ is > 0. Let $r \geqslant 0$ be such that $f(r) > \epsilon^{-1}$. Then for all $n \geqslant$ some $n_0 \geqslant 0$, we have $p_n(r) > \epsilon^{-1}$. Hence if $n \geqslant n_0$ and $x > r$, we have $|[f(x)]^{-1} - [p_n(x)]^{-1}| < 1/p_n(x) \leqslant 1/p_n(r) < \epsilon$. Let $n_1 \geqslant n_0$ be such that if $0 \leqslant x \leqslant r$ and $n \geqslant n_1$, we have $f(x) - p_n(x) < \epsilon f^2(0)$. If $n \geqslant n_1$ and $0 \leqslant x \leqslant r$, then

$$\begin{aligned} &|[f(x)]^{-1} - [p_n(x)]^{-1}| \\ &\quad = [f(x) - p_n(x)]/[f(x)p_n(x)] \leqslant [f(x) - p_n(x)]/f^2(0) < \epsilon. \end{aligned}$$

Hence $\sup_{0\leqslant x<\infty} |[f(x)]^{-1} - [p_n(x)]^{-1}| < \epsilon$ if $n \geqslant n_1$.

Necessity. Let $0 < r < \infty$. Let $n_2 \geqslant 0$ be such that $\sup_{0\leqslant x<\infty} |[f(x)]^{-1} - [p_n(x)]^{-1}| < [2 \max_{0\leqslant t\leqslant r} f(t)]^{-1}$ whenever $n \geqslant n_2$. For such an n, if $0 \leqslant x \leqslant r$, then $[p_n(x)]^{-1} > [f(x)]^{-1} - [2 \max_{0\leqslant t\leqslant r} f(t)]^{-1} \geqslant [2f(x)]^{-1}$, and hence $|f(x) - p_n(x)| = f(x)p_n(x)| [f(x)]^{-1} - [p_n(x)]^{-1}| \leqslant 2f^2(x)| [f(x)]^{-1} - [p_n(x)]^{-1}|$. Therefore if $n \geqslant n_2$, then $\max_{0\leqslant x\leqslant r} |f(x) - p_n(x)| \leqslant 2 \max_{0\leqslant x\leqslant r} f^2(x) \cdot \sup_{0\leqslant x<\infty} | [f(x)]^{-1} - [p_n(x)]^{-1}| \to 0$ as $n \to \infty$. Thus $p_n(x)$ converges uniformly to f in $[0, r]$. As the coefficients of each $p_n(x)$ are $\geqslant 0$, there are $a_0, a_1, a_2,\ldots$, all $\geqslant 0$, such that $f(x) = \sum_{k=0}^{\infty} a_k x^k$ throughout $(0, r)$ ([3, p. 154]; for a very elementary proof see [2]). Since $r > 0$ is arbitrary, the result follows.

References

1. A. R. Reddy and O. Shisha, A class of rational approximations on the positive real axis—A survey, *J. Approximation Theory* **12** (1974), 425–434.
2. O. Shisha, On sequences of power series with restricted coefficients, *Amer. Math. Monthly* **72** (1965), 533–537.
3. D. V. Widder, "The Laplace Transform," Princeton University Press, 1946.

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